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ANTIPLANE PROBLEMS IN COMPOSITE ANISOTROPIC MATERIALS WITH AN INCLINED CRACK TERMINATING AT A BIMATERIAL INTERFACE

CHIEN-CHING MA and BAO-LUH HOUR
Department of Mechanical Engineering, National Taiwan University, Taipei, Taiwan 10764, Republic of China

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Abstract—The antiplane strain problem for bonded dissimilar half planes of general anisotropic material containing an inclined crack terminating at the bimaterial interface is considered. The surfaces of the crack can be subjected to traction–traction, traction–displacement or displacement–displacement boundary conditions. The dependence of the order of the stress singularity on the inclined angle and material constants is studied. If the effective crack angle and the effective material constant are introduced for the anisotropic case, then the characteristic equation which determines the order of the stress singularity has the same functional form as the isotropic case. Explicit solutions for the order of stress singularity are obtained for some special cases. It is found that the order of the stress singularity is always real for all the cases studied in this paper. This is a quite different character from the in-plane case in which the complex type of stress singularity might exist. The angular distribution of stresses near the crack tip and the exact full field stress solutions are also investigated.

1. INTRODUCTION

Problems related to stress singularities have received much attention, especially if cracked geometries are included. The appearance of flaws or cracks on the bond between the two materials could reduce the strength of the structure significantly and Williams (1959) was the first to consider this problem. He found that the stresses are inversely proportional to the square root of the radial distance from the crack tip and possess a sharp oscillatory character near the crack tip. This problem was further addressed by Erdogan (1963), England (1965), Erdogan (1965) and Rice and Sih (1965). Tranter (1948) used the Mellin transform in conjunction with the Airy stress function representation of plane elasticity to solve the isotropic wedge problem. Bogy (1971a) used the Mellin transform to treat the problem of two materially dissimilar isotropic elastic wedges of arbitrary angles that are bonded together along a common edge. A number of other workers have studied similar problems, see Dempsey and Sinclair (1981) and Erdogan and Gupta (1972), for example.

Extensions to anisotropic materials have been made by Sih *et al.* (1965), Gotoh (1967) and Willis (1971). Following the approach of Stroh (1958, 1962), Ting and Chou (1981) and Ting (1986) studied the stress distribution near the composite wedge of anisotropic materials. Bogy (1972), Kuo and Bogy (1974a, b) employed a complex function representation of the solution (Green and Zerna, 1954) in conjunction with a generalized Mellin transform to analyze stress singularities in an anisotropic wedge. From the recent study, Ma and Hour (1989) found that the order of stress singularity is always real for general anisotropic bimaterial wedges of antiplane problem. Several studies in this area have been made in the last decade, see Clements (1971), Delale and Erdogan (1979), Hoenig (1982) and Wang and Choi (1982a, b).

The strength of composite materials is influenced by the orientation of existing cracks with respect to the bimaterial interface. When a crack encounters an interface with a second material, it may be penetrated through the interface, it may be reflected back, or interfacial debonding may occur. Under some situations an interface can provide a mechanism for crack arrest. In this paper, antiplane strain problems of general anisotropic dissimilar material containing an inclined crack terminating at the interface are considered. The

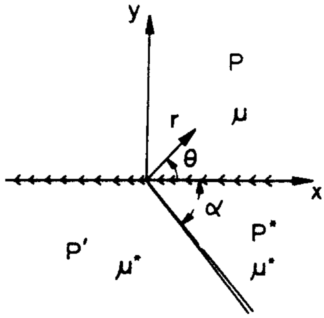

Fig. 1. Configuration of crack terminating at the interface.

problem under consideration is the generalization of that considered by Bassani and Erdogan (1979) in which only the case of isotropic dissimilar material under traction boundary condition was studied. Here the problem for anisotropic material of traction (or displacement) prescribed on both crack faces, and the problem of traction prescribed on one face with displacement prescribed on the other are solved. The correspondent in-plane problem of isotropic material subjected to traction–traction boundary condition was examined by Bogy (1971b). The problem of a straight crack which is perpendicularly terminating at and passing through a bimaterial interface of isotropic and orthotropic half planes was studied by Kasano *et al.* (1986, 1987). The problem will be solved by application of Mellin transform as done by Ma and Hour (1989). We focus our attention especially on the dependence of the order of the stress singularity on the inclined angle, material constants and boundary conditions. The angular dependence of the stress field near the crack tip and the full field stresses distribution are also analyzed. Unlike the existence of the oscillatory character for the singular behavior of the in-plane problem, we found that the order of stress singularity is real for all the cases studied in the antiplane strain problem subjected to different boundary conditions (i.e. traction–traction, displacement–displacement, traction–displacement). Furthermore, if "effective angle" and "effective material constant" are introduced, then the order of the singularity for the general anisotropic material can be obtained easily from the solution of the isotropic case.

2. GENERAL SOLUTION IN MELLIN TRANSFORM DOMAIN

Let P, P^* and P' denote the open two-dimensional regions occupied by the cross-sections of a half space and its adjacent half space split by a plane crack that subtends an angle α ($\alpha \leqslant \pi$) with the interface as shown in Fig. 1. Let μ and μ^* stand for the shear moduli of the two different media in regions $0 \leqslant \theta \leqslant \pi$ and $\pi \leqslant \theta \leqslant 2\pi$. Two dissimilar elastic homogeneous materials are assumed to be perfectly bonded along the interface. For the antiplane shear deformation, the only nonvanishing displacement component is along the z-axis, $w(x, y)$. In the absence of body forces, the equilibrium equation for w is given by the Laplace equation

$$\nabla^2 w = 0. \tag{1}$$

Using the relations between the shear stress and displacement, the nonvanishing stresses are

$$\tau_{rz} = \mu \frac{\partial w}{\partial r}, \tag{2}$$

$$\tau_{\theta z} = \frac{\mu}{r} \frac{\partial w}{\partial \theta}. \tag{3}$$

In addition, we shall require the stress fields to satisfy the regularity conditions,

$$\tau_{rz}, \tau_{\theta z} = O(r^{-1+\delta}) \quad \text{as} \quad r \to \infty \quad \text{for} \quad \delta > 0. \tag{4}$$

Let the Mellin transform of a function $f(r)$ be denoted by $\hat{f}(s)$

$$\hat{f}(s) = M\{f; s\} = \int_0^\infty f(r) r^{s-1}\, dr, \tag{5}$$

where s is a complex transform parameter. The Mellin transforms of $w(r, \theta)$, $r\tau_{rz}(r, \theta)$, $r\tau_{\theta z}(r, \theta)$ with respect to r are denoted by $\hat{w}(s, \theta)$, $\hat{\tau}_{rz}(s, \theta)$ and $\hat{\tau}_{\theta z}(s, \theta)$. Thus

$$\hat{w}(s, \theta) = \int_0^\infty w(r, \theta) r^{s-1}\, dr, \tag{6}$$

$$\hat{\tau}_{rz}(s, \theta) = \int_0^\infty \tau_{rz}(r, \theta) r^{s}\, dr, \tag{7}$$

$$\hat{\tau}_{\theta z}(s, \theta) = \int_0^\infty \tau_{\theta z}(r, \theta) r^{s}\, dr. \tag{8}$$

By use of the inversion theorem for the Mellin transform, the stress and displacement components are given by

$$w(r, \theta) = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \hat{w}(s, \theta) r^{-s}\, ds, \tag{9}$$

$$\tau_{rz}(r, \theta) = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \hat{\tau}_{rz}(s, \theta) r^{-s-1}\, ds, \tag{10}$$

$$\tau_{\theta z}(r, \theta) = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \hat{\tau}_{\theta z}(s, \theta) r^{-s-1}\, ds. \tag{11}$$

Because of condition (4), the path of integration in the complex line integrals $\mathrm{Re}(s) = \rho$ in (9), (10) and (11) must lie within a common strip of regularity of their integrands, the choice of ρ is taken to be

$$\rho = -\varepsilon, \quad 0 < \varepsilon < (|\mathrm{Re}(s_1)|), \tag{12}$$

where s_1 denotes the location of the pole in the open strip $-1 < \mathrm{Re}(s) < 0$ with the largest real part and Re denotes the real part of the complex argument.

Applying the Mellin transform (6) to (1) yields an ordinary differential equation for $\hat{w}$, the general solution of which is

$$\hat{w}(s, \theta) = a(s) \sin(s\theta) + b(s) \cos(s\theta). \tag{13}$$

The solutions of the stress components in the transformed form appear as

$$\hat{\tau}_{rz}(s, \theta) = -\mu s[a(s) \sin(s\theta) + b(s) \cos(s\theta)], \tag{14}$$

$$\hat{\tau}_{\theta z}(s, \theta) = \mu s[a(s) \cos(s\theta) - b(s) \sin(s\theta)]. \tag{15}$$

3. THE STRESS SINGULARITIES AT THE CRACK TIP

Case I. *Traction–traction boundary condition*

Perfect bonding along the interface is ensured by the stress and displacement continuity conditions, and the traction boundary conditions on the crack faces are given as follows

$$\begin{aligned}\tau^*_{\theta z}(r,-\alpha) &= t^*(r), \quad \tau'_{\theta z}(r,2\pi-\alpha) = t'(r),\\ \tau^*_{\theta z}(r,0) &= \tau_{\theta z}(r,0), \quad w^*(r,0) = w(r,0),\\ \tau'_{\theta z}(r,\pi) &= \tau_{\theta z}(r,\pi), \quad w'(r,\pi) = w(r,\pi).\end{aligned} \tag{16}$$

In (16), $t^*(r)$ and $t'(r)$ represent the shearing tractions prescribed on the crack faces. Substitution of (13)–(15) into the Mellin transform of (16) provides the following six equations for the six unknown functions $a^*(s)$, $b^*(s)$, $a'(s)$, $b'(s)$, $a(s)$, $b(s)$;

$$\begin{gathered}\mu^*a^*(s)\cos(\alpha s)+\mu^*b^*(s)\sin(\alpha s) = \hat{t}^*(s)/s,\\ \mu^*a'(s)\cos(s(2\pi-\alpha))-\mu^*b'(s)\sin(s(2\pi-\alpha)) = \hat{t}'(s)/s,\\ \mu^*a^*(s)-\mu a(s) = 0,\\ b^*(s)-b(s) = 0,\\ \mu^*a'\cos(s\pi)-\mu^*b'\sin(s\pi)-\mu a\cos(s\pi)+\mu b\sin(s\pi) = 0,\\ a'\sin(s\pi)+b'\cos(s\pi)-a\sin(s\pi)-b\cos(s\pi) = 0,\end{gathered} \tag{17}$$

where $\hat{t}^*(s)$ and $\hat{t}'(s)$ denote the Mellin transforms of $t^*(r)$ and $t'(r)$, respectively. The solution of (17) together with (13)–(15) determine the exact solutions of stresses $\hat{\tau}^*_{rz}(s,\theta)$, $\hat{\tau}^*_{\theta z}(s,\theta)$, $\hat{\tau}'_{rz}(s,\theta)$, $\hat{\tau}'_{\theta z}(s,\theta)$ and $\hat{\tau}_{rz}(s,\theta)$, $\hat{\tau}_{\theta z}(s,\theta)$ in the transformed form

$$\hat{\tau}^*_{rz}(s,\theta) = \frac{2}{(1+R)D}\{-[(RS_3+C_3)\hat{t}^*+RS_1\hat{t}']\sin s\theta+[(R^2S_4-RC_4)\hat{t}^*+RC_1\hat{t}']\cos s\theta\}, \tag{18}$$

$$\hat{\tau}^*_{\theta z}(s,\theta) = \frac{2}{(1+R)D}\{[(RS_3+C_3)\hat{t}^*+RS_1\hat{t}']\cos s\theta+[(R^2S_4-RC_4)\hat{t}^*+RC_1\hat{t}']\sin s\theta\}, \tag{19}$$

$$\hat{\tau}'_{rz}(s,\theta) = \frac{2}{(1+R)D}\{-[RS_2\hat{t}^*+(-RS_3+\Phi)\hat{t}']\sin s\theta-[RC_2\hat{t}^*-(RC_4+\Psi)\hat{t}']\cos s\theta\}, \tag{20}$$

$$\hat{\tau}'_{r\theta}(s,\theta) = \frac{2}{(1+R)D}\{[RS_2\hat{t}^*+(-RS_3+\Phi)\hat{t}']\cos s\theta-[RC_2\hat{t}^*-(RC_4+\Psi)\hat{t}']\sin s\theta\}, \tag{21}$$

$$\hat{\tau}_{rz}(s,\theta) = \frac{2}{(1+R)D}\{-[(RS_3+C_3)\hat{t}^*+RS_1\hat{t}']\sin s\theta+[(RS_4-C_4)\hat{t}^*+C_1\hat{t}']\cos s\theta\}, \tag{22}$$

$$\hat{\tau}_{r\theta}(s,\theta) = \frac{2}{(1+R)D}\{[(RS_3+C_3)\hat{t}^*+RS_1\hat{t}']\cos s\theta+[(RS_4-C_4)\hat{t}^*+C_1\hat{t}']\sin s\theta\}, \tag{23}$$

in which $R = \mu^*/\mu$ and

$$D(\alpha, R; s) = \sin s\pi[(1-R)\cos(2\alpha-\pi)s+(1+R)\cos s\pi], \tag{24}$$

$$C_1 = \cos\alpha s, \quad S_1 = \sin\alpha s,$$

$$C_2 = \cos(2\pi-\alpha)s, \quad S_2 = \sin(2\pi-\alpha)s,$$

$$C_3 = \sin s\pi\cos(\pi-\alpha)s, \quad S_3 = \cos s\pi\sin(\pi-\alpha)s,$$

$$C_4 = \cos s\pi\cos(\pi-\alpha)s, \quad S_4 = \sin s\pi\sin(\pi-\alpha)s,$$

$$\Phi = \sin s\pi(\cos s\pi\cos\alpha s+R^2\sin s\pi\sin\alpha s),$$

$$\Psi = \sin s\pi(\sin s\pi\cos\alpha s-R^2\cos s\pi\sin\alpha s).$$

From (18) to (23), it is clear that $\hat{\tau}_{rz}(s, \theta)$, $\hat{\tau}_{\theta z}(s, \theta)$ etc., are meromorphic functions of s for fixed θ in $-1 < \mathrm{Re}(s) < 0$ whose poles can occur only at the zeros of $D(s)$ in the open strip. We can now indicate the appropriate path of integration for the inversion integrals in (10) and (11). We may then choose the path of integration for the inversion integrals to lie within the common strip of regularity $\mathrm{Re}(s_1) < \rho < 0$ with s_1 denoting the zero of $D(s)$ with the largest real part in the strip.

If s_1 is a simple zero of $D(s)$, then the type of singularity will be of the order $\lambda = \mathrm{Re}(s_1)+1$. Evidently if s_1 is a complex zero, then the stress fields are oscillatory in the limit $r \to 0$. If no zero of $D(s)$ occurs in $-1 < \mathrm{Re}(s) < 0$, but $\mathrm{d}D(s)/\mathrm{d}s = 0$ at $s = -1$, then it will have logarithm type singularity. Hence, determination of the location of the zeros of the characteristic function $D(s)$ in the strip $-1 < \mathrm{Re}(s) < 0$ is our principal task. It is shown in the Appendix that the zeros of $D(s)$ are always real for any combination of material constants and crack inclined angle α, so that the possibility of the oscillatory singular behavior is precluded.

We examine $D(s)$ for various special cases to get explicit analytical results. When $\alpha = 0$ (or $\alpha = \pi$), the problem becomes that of two dissimilar materials with crack along their common interface and the familiar square root singularity is obtained. Suppose that the materials occupying P^* and P are the same, that is $R = 1$, we also have the well-known result of square root singularity. If the material in P^* is infinitely rigid, i.e. $R \to \infty$, we have $\lambda \to 1$. If the material in P is infinitely rigid, i.e. $R \to 0$, we have

$$\lambda = 1-\frac{\pi}{2}\left(\frac{1}{\pi-\alpha}\right) \quad \text{for} \quad 0 \leqslant \alpha \leqslant \pi/2,$$

$$\lambda = 1-\frac{\pi}{2\alpha} \quad \text{for} \quad \pi/2 \leqslant \alpha \leqslant \pi.$$

For the case of a crack perpendicular to the interface ($\alpha = \pi/2$), we have

$$\lambda = 1-\frac{1}{\pi}\tan^{-1}\left(\frac{2\sqrt{R}}{R-1}\right) \quad \text{for} \quad R \geqslant 1,$$

$$\lambda = \frac{1}{\pi}\tan^{-1}\left(\frac{2\sqrt{R}}{R-1}\right) \quad \text{for} \quad R < 1.$$

For $\alpha = \pi/4$, we have

$$\lambda = 1-\frac{2}{\pi}\cos^{-1}\left[\frac{R-1+\sqrt{9R^2+14R+9}}{4(1+R)}\right]$$

We now turn to the numerical computation of the zeros of $D(s)$ for general cases. The

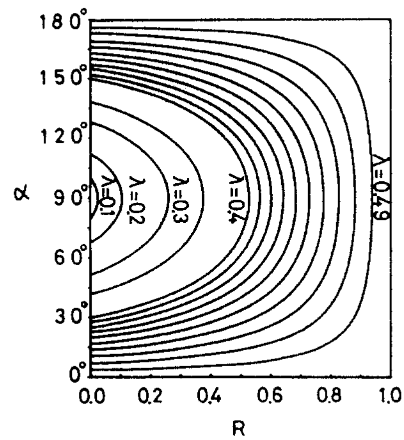

Fig. 2. Dependence of the order of stress singularity λ on α and $R (R \leqslant 1)$ for the traction–traction boundary condition.

results of the numerical computations are given in Figs 2 and 3, which show the dependence of the order of stress singularity λ on $R = \mu^*/\mu$ and α. When curves corresponding to different values of λ overlap, i.e. when multiple roots occur in $0 < \lambda < 1$, it is understood that the larger value of λ is plotted which controlled the asymptotic stress as $r \to 0$. It shows that the order of the stress singularity is symmetric with respect to $\alpha = 90°$. The order of stress singularity for $R < 1$ $(\mu^* < \mu)$ will be in the range $0 < \lambda < 0.5$, which is smaller than the case for $R > 1$ $(0.5 < \lambda < 1)$. This indicates that the stress near the crack tip will be more singular if the crack occurs in the region of large shear modulus material. The angular dependence of stresses near the crack tip for $\alpha = 75°$ and $R = 0.05, 0.1, 0.4$ are plotted in Fig. 4. The shear stress $\tau_{\theta z}$ is continuous on the bonded edge while τ_{rz} is discontinuous at the interface.

The exact full field shear stresses of τ_{rz} and $\tau_{\theta z}$ are computed numerically for $\alpha = 30°$, $45°$ and $R = 2$. The specific loading considered here is that of a uniform shear stress $\tau_{\theta z}$ with unit magnitude applied from $r = 0$ to $r = 1$ on the crack faces. Thus, the load functions on the boundary will be

$$t^*(r) = H(1-r), \quad t'(r) = -H(1-r),$$

where H is the Heaviside function. The results of the computations of stresses along the interface $\theta = 0$ are shown in Fig. 5.

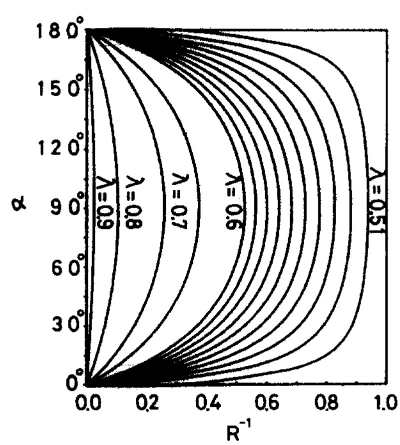

Fig. 3. Dependence of the order of stress singularity λ on α and $R (R \geqslant 1)$ for the traction–traction boundary condition.

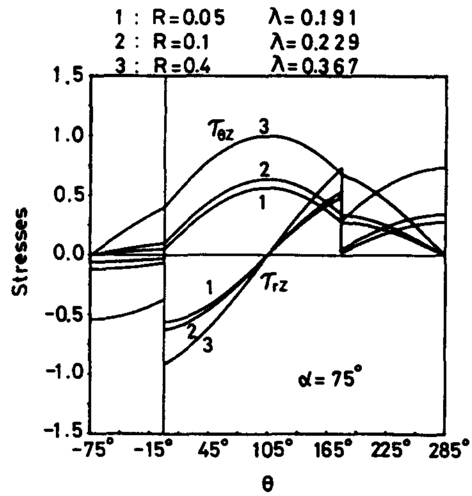

Fig. 4. The angular distribution of stresses $\tau_{\theta z}$ and τ_{rz} of the asymptotic behavior as $r \to 0$ for the traction–traction boundary condition.

Case II. *Traction–displacement boundary condition*

Here the problem of traction prescribed on one crack face with displacement prescribed on the other is solved. The continuity conditions of stress and displacement along the interface are the same as that expressed in traction–traction boundary condition. Thus we consider the following boundary conditions on the crack faces,

$$w^*(r, -\alpha) = W^*(r),$$
$$\tau'_{\theta z}(r, 2\pi-\alpha) = t'(r). \tag{25}$$

The solutions presented follow the outline established previously. Here we only concentrate our attention on the order of stress singularity. The characteristic equation which determines the order of stress singularity is

$$D(\alpha, R; s) = (1+R)\sin^2(s\pi)+(R-1)\sin(s\pi)\sin[s(\pi-2\alpha)]-\frac{2R}{R+1} = 0. \tag{26}$$

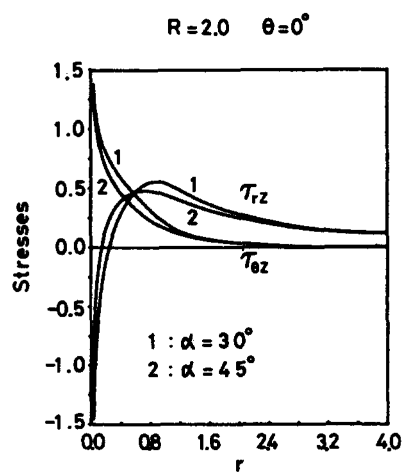

Fig. 5. Stresses $\tau_{\theta z}$ and τ_{rz} along the interface $\theta = 0$ for $R = 2.0$ and $\alpha = 30°, 45°$.

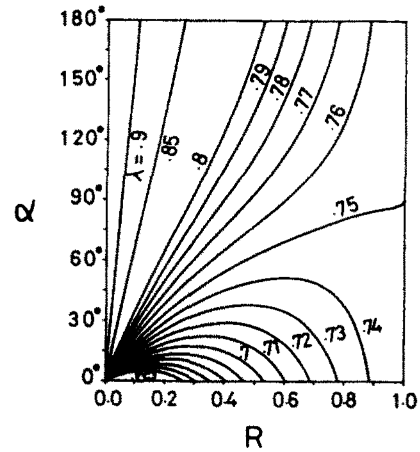

Fig. 6. Dependence of the order of stress singularity λ on α and $R(R \leqslant 1)$ for the traction–displacement boundary condition.

For the case of interfacial crack, the order of stress singularity will be

$$\lambda = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\sqrt{R} \quad \text{for} \quad \alpha = 0,$$

$$\lambda = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\sqrt{\frac{1}{R}} \quad \text{for} \quad \alpha = \pi. \tag{27}$$

For the case of a crack perpendicular to the interface ($\alpha = \pi/2$), we have

$$\lambda = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\sqrt{\frac{1+R^2}{2R}}. \tag{28}$$

If the two materials are the same ($R = 1$), then we have the familiar value of λ equals to 3/4. For both cases of $R \to 0$ and $R \to \infty$, we all have $\lambda \to 1$ which is the largest value of λ and hence the most severe stress singularity. The results of the numerical calculations for the general cases of the order of stress singularity are shown in Figs 6 and 7. Furthermore, (26) does not change by interchanging R by $1/R$ and α by $\pi - \alpha$ and Figs 6 and 7 also show this feature.

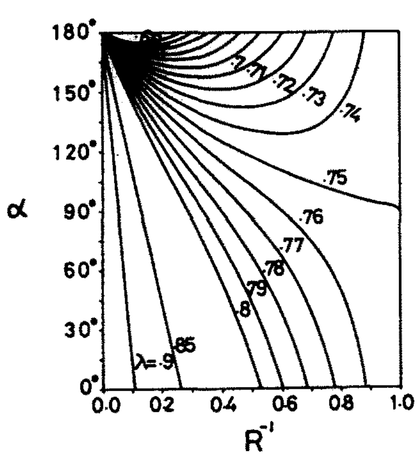

Fig. 7. Dependence of the order of stress singularity λ on α and $R(R \geqslant 1)$ for the traction–displacement boundary condition.

Case III. *Displacement–displacement boundary condition*

We consider displacements prescribed at the crack faces $\theta = 2\pi - \alpha$ and $\theta = -\alpha$ of the form,

$$w^*(r, -\alpha) = W^*(r),$$
$$w'(r, 2\pi - \alpha) = W'(r). \tag{29}$$

The characteristic equation will be

$$D(\alpha, R; s) = (1+R)\cos(s\pi) + (R-1)\cos[s(2\alpha - \pi)] = 0. \tag{30}$$

Equation (30) has exactly the same form as (24) for the prescribed traction condition except replacing R by $1/R$.

4. STRESS SINGULARITIES FOR ANISOTROPIC MATERIAL

In this section, the problem for two dissimilar anisotropic materials containing an inclined crack with angle α, is formulated. The method employs the complex representation of the antiplane anisotropic elasticity solution in conjunction with a generalization of the Mellin transform. Attention is also focused on the dependence of the order of the power singularities in the stress field at the crack angle and material constants. If the plane of elastic symmetry is assumed to be normal to the z-axis, then there are only three relevant coefficients c_{44}, c_{45} and c_{55} to be considered. The stress components are related to the displacement as follows

$$\tau_{yz} = c_{44}\frac{\partial w}{\partial y} + c_{45}\frac{\partial w}{\partial x}, \tag{31}$$

$$\tau_{xz} = c_{45}\frac{\partial w}{\partial y} + c_{55}\frac{\partial w}{\partial x}. \tag{32}$$

The corresponding displacement equation of equilibrium is

$$c_{55}\frac{\partial^2 w}{\partial x^2} + 2c_{45}\frac{\partial^2 w}{\partial x \partial y} + c_{44}\frac{\partial^2 w}{\partial y^2} = 0. \tag{33}$$

The governing eqn (33) can be solved in the complex plane $z = x + py$ such that

$$w(x, y) = 2\mathrm{Re}[U(z)], \tag{34}$$

where U is an arbitrary function of z and p is a value dependent on the elasticity constants. Substitution of (34) into (33) yields p which must satisfy the following equation

$$c_{44}p^2 + 2c_{45}p + c_{55} = 0, \tag{35}$$

hence

$$p = \frac{-c_{45} \pm i\sqrt{c_{44}c_{55} - (c_{45})^2}}{c_{44}}$$

It is expedient to define

$$\phi(z) = i\sqrt{c_{44}c_{55}-(c_{45})^2}\frac{\mathrm{d}U}{\mathrm{d}z}, \tag{36}$$

so that the shear stresses may be written simply as

$$\tau_{xz} = -(p\phi+\bar{p}\bar{\phi}), \tag{37}$$

$$\tau_{yz} = \phi+\bar{\phi}, \tag{38}$$

where overline denotes complex conjugate. Consider the stress transformation

$$\tau_{\theta z} = \tau_{yz}\cos\theta-\tau_{xz}\sin\theta, \tag{39}$$

$$\tau_{rz} = \tau_{yz}\sin\theta+\tau_{xz}\cos\theta. \tag{40}$$

The solution of the problem is obtained by the use of the integral transform which is a complex analogy of the standard Mellin transform. Let $\hat{U}(s)$ be defined by

$$\hat{U}(s) = \int_0^\infty U(z)z^{s-1}\,\mathrm{d}z = (\cos\theta+p\sin\theta)^s\int_0^\infty U(z)r^{s-1}\,\mathrm{d}r, \tag{41}$$

in which the path of integration is along a ray of fixed θ and s is a complex transform parameter. We obtain also from the conjugate of (41)

$$\hat{\bar{U}}(s) = \int_0^\infty \bar{U}(\bar{z})\bar{z}^{s-1}\,\mathrm{d}\bar{z} = (\cos\theta+\bar{p}\sin\theta)^s\int_0^\infty \bar{U}(\bar{z})r^{s-1}\,\mathrm{d}r.$$

From a formal integration by parts and with appropriately assumed behavior as $r\to 0$ and ∞, we have

$$\int_0^\infty U'(z)r^s\,\mathrm{d}r = -\frac{s\hat{U}(s)}{(\cos\theta+p\sin\theta)^{s+1}}, \tag{42}$$

$$\int_0^\infty \bar{U}'(\bar{z})r^s\,\mathrm{d}r = -\frac{s\hat{\bar{U}}(s)}{(\cos\theta+\bar{p}\sin\theta)^{s+1}}. \tag{43}$$

If the integral operation is applied to (34) and (39) and use is made of (41)–(43) there follows

$$\hat{\tau}_{\theta z}(s,\theta) = -iC\left[\frac{s\hat{U}(s)}{H(\theta)}-\frac{s\hat{\bar{U}}(s)}{\bar{H}(\theta)}\right], \tag{44}$$

$$\hat{w}(s,\theta) = \frac{\hat{U}(s)}{H(\theta)}+\frac{\hat{\bar{U}}(s)}{\bar{H}(\theta)}, \tag{45}$$

in which

$$C = [c_{44}c_{55}-(c_{45})^2]^{1/2}, \tag{46}$$
$$H(\theta) = (\cos\theta+p\sin\theta)^s.$$

In the same definition as the isotropic case, $\hat{\tau}_{\theta z}(s,\theta)$ is the Mellin transform with respect to r of $r\tau_{\theta z}(r, \theta)$. The traction prescribed boundary conditions as shown in (16) in conjunction with (44) and (45) yield for the determination of the six unknowns $\hat{U}(s)$, $\hat{U}^*(s)$, etc. for the following inhomogeneous system of six equations,

$$\begin{aligned}
&C^*\hat{U}^* - C^*\hat{\bar{U}}^* - C\hat{U} + C\hat{\bar{U}} = 0,\\
&\hat{U}^* + \hat{\bar{U}}^* - \hat{U} - \hat{\bar{U}} = 0,\\
&\frac{\hat{U}^*}{H(-\alpha)} - \frac{\hat{\bar{U}}^*}{\bar{H}(-\alpha)} = \frac{\hat{t}^*(s)}{-iC^*s},\\
&\frac{\hat{U}'}{H(2\pi-\alpha)} - \frac{\hat{\bar{U}}'}{\bar{H}(2\pi-\alpha)} = \frac{\hat{t}'(s)}{-iC^*s},\\
&C'\left[\frac{\hat{U}'}{H(\pi)} - \frac{\hat{\bar{U}}'}{\bar{H}(\pi)}\right] - C\left[\frac{\hat{U}}{H(\pi)} - \frac{\hat{\bar{U}}}{\bar{H}(\pi)}\right] = 0,\\
&\frac{\hat{U}'}{H(\pi)} + \frac{\hat{\bar{U}}'}{\bar{H}(\pi)} - \frac{\hat{U}}{H(\pi)} - \frac{\hat{\bar{U}}}{\bar{H}(\pi)} = 0.
\end{aligned} \tag{47}$$

This system can be solved and the expressions for $\hat{\tau}_{\theta z}(s, \theta)$, $\hat{w}(s, \theta)$ now follow directly from substituting the solution of (47) into (44) and (45). This completes the formal solution for the transforms of the stress and displacement components. As discussed in the isotropic material case in the previous section, the dependence of the order of the stress field singularity on crack angle α and material parameters is determined by the pole of the meromorphic function $\hat{\tau}_{\theta z}(s, \theta)$, etc. or the location of the zero of the following characteristic equation

$$(1-Q)\cos(2\xi-\pi)s + (1+Q)\cos s\pi = 0, \tag{48}$$

where

$$Q = \frac{\sqrt{c^*_{44}c^*_{55} - (c^*_{45})^2}}{\sqrt{c_{44}c_{55} - (c_{45})^2}}, \tag{49}$$

$$\tan\xi = \frac{\sqrt{c^*_{44}c^*_{55} - (c^*_{45})^2}\sin\alpha}{c^*_{44}\cos\alpha + c^*_{45}\sin\alpha}. \tag{50}$$

It is surprising that (48) has exactly the same functional form as (24) for the isotropic case. Here ξ is called the effective angle that is defined in (50), Q is the ratio of two anisotropic material constants and is defined in (49). This result shows that the problem of solving the anisotropic bimaterial inclined crack can be simplified as the isotropic case. If we rotate the inclined crack angle α to effective angle ξ, and define the material parameter C in (46) as the effective material constant which will be equivalent to shear modulus μ in the isotropic case, then the anisotropic problem can be analyzed as the isotropic case. If the effective angle ξ is defined, then the order of stress singularity for anisotropic bimaterial inclined crack depends only on one material parameter, the ratio of two effective material constants, instead of six anisotropic material constants. For the isotropic case, $c_{45} = 0$ and $c_{44} = c_{55} = \mu$, we have $Q = \mu^*/\mu = R$ and $\xi = \alpha$, then (48) reduces to the isotropic case as shown in (24). For the interfacial crack problem, $\alpha = 0$ (or π), we have the effective angle $\xi = 0$ (or π) so that the interfacial crack in the general anisotropic material of antiplane problem also gives rise to the square root singularity and is independent of the material constants. Because the order of singularity in the present case shares the same feature as that in the isotropic case, the discussion will not be repeated here. But it is worthy of mentioning again that the order of the singularity for the anisotropic case is real and the oscillatory singular behavior is not presented.

The problem of traction prescribed on one face with displacement prescribed on the other as shown in the boundary condition (25) can be analyzed in a similar way. The result is

$$(1+Q)\sin^2(s\pi)+(Q-1)\sin s\pi \sin[s(\pi-2\xi)]-\frac{2Q}{Q+1}=0. \quad (51)$$

Again, (51) has exactly the same functional form as (26) for Q and ξ defined in (49)–(50).

Finally, for the prescribed displacements at both boundary faces as indicated in (29), the order of the stress singularity is obtained from solving the following equation

$$(1+Q)\cos(s\pi)+(Q-1)\cos[s(2\xi-\pi)]=0. \quad (52)$$

5. CONCLUDING REMARKS

The problem of antiplane shear for anisotropic dissimilar materials with crack terminating at the interface was solved by a straightforward application of the Mellin transform. Emphasis is placed on the investigation of the order of stress singularity and the angular dependence in the stress field at the crack tip. It is shown in this paper that the order of the stress singularity λ is always real for general anisotropic material for various boundary conditions. This is a quite different character from the in-plane case in which λ may be complex. If effective angle and material constant are introduced in the analysis for the anisotropic case, then the characteristic equation which determines the order of stress singularity, has the same functional form as that for isotropic case. These results may simplify the analysis for the anisotropic problem. It is worthy to note that if the effective inclined crack angle is defined, then the order of stress singularity depends only on one material parameter instead of six material constants for the anisotropic bimaterial inclined crack problem.

Explicit solutions of the order of stress singularity were obtained for some special cases, interfacial crack ($\alpha = 0$ or π) and crack perpendicular to the interface problems ($\alpha = \pi/2$). It shows that the familiar square root singularity is obtained for the dissimilar anisotropic materials with crack in the interface of traction–traction and displacement–displacement boundary conditions, while for traction–displacement boundary conditions, the order of stress singularity of dissimilar anisotropic materials with cracks in the interface will depend on material constants. The result shows that the stress near the crack tip will be more singular if the crack occurs in the high shear modulus material for traction–traction boundary conditions and in the low shear modulus material for displacement–displacement boundary conditions. For traction–traction boundary conditions, the crack tip will be more singular if a crack occurs in the interface for $R < 1$, but for $R > 1$, the crack tip will be more singular if the crack is perpendicular to the interface.

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APPENDIX

From eqn (24)

$$\frac{\cos(s\pi)}{\cos(2\alpha-\pi)s}=\frac{R-1}{R+1}, \tag{A1}$$

we have

$$\cos(s\pi)=\Omega\cos(2\alpha-\pi)s, \tag{A2}$$

where

$$-1\leqslant\Omega=\frac{1-R}{1+R}\leqslant 1.$$

Assume (A2) has the complex root of the form $s=x+iy$ and $x\neq 0$, $y\neq 0$, then (A2) can be rewritten as follows

$$\cos(\pi x)\cosh(\pi y)-i\sin(\pi x)\sinh(\pi y)=\Omega\cos(2\alpha-\pi)x\cosh(2\alpha-\pi)y-i\Omega\sin(2\alpha-\pi)x\sinh(2\alpha-\pi)y. \tag{A3}$$

Equating the real and imaginary parts of (A3) yields

$$\cos(\pi x)\cosh(\pi y)=\Omega\cos(2\alpha-\pi)x\cosh(2\alpha-\pi)y, \tag{A4}$$

$$\sin(\pi x)\sinh(\pi y)=\Omega\sin(2\alpha-\pi)x\sinh(2\alpha-\pi)y. \tag{A5}$$

(A4) and (A5) can be combined into the following equation

$$\cos^2(\pi x)\left\{\frac{\cosh(\pi y)}{\cosh(2\alpha-\pi)y}\right\}^2+\sin^2(\pi x)\left\{\frac{\sinh(\pi y)}{\sinh(2\alpha-\pi)y}\right\}^2=\Omega^2. \tag{A6}$$

Since $|2\alpha-\pi|<\pi$, hence

$$\left\{\frac{\cosh(\pi y)}{\cosh(2\alpha-\pi)y}\right\}^2>1,$$

and

$$\left\{\frac{\sinh(\pi y)}{\sinh(2\alpha-\pi)y}\right\}^2>1,$$

which make the left-hand side of (A6) greater than 1. But the right-hand side of (A6) is always less than 1 and we have a contradiction. If $x = 0$, and $y \neq 0$, from (A3) we also get a contradictory result. Hence the only possibility to find the solution of (A2) is $x \neq 0$, $y = 0$ which indicates that the order of stress singularity is real and this completes the proof.